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How Cold Dark Matter Theory Explains Milgrom’s Law

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ABSTRACT

Milgrom noticed the remarkable fact that the gravitational effect of dark matter in galaxies only becomes important where accelerations are less than about $10^{-8} \text{ cm s}^{-2} \sim cH_0$. This forms the basis for his Modified Newtonian Dynamics (MOND), an alternative to particle dark matter. We show that in the Cold Dark Matter (CDM) theory of structure formation Milgrom’s Law comes about automatically, owing to the scale-free character of the density perturbations, baryonic dissipation and numerical coincidences. With the evidence for CDM mounting, and at the same time problems for MOND becoming more numerous, it appears that Milgrom’s Law is an accident rather than an important clue to the dark-matter puzzle.

Subject headings: cosmology: dark matter—galaxies: dynamics

1. Introduction

The dark-matter mystery has been with us since Zwicky noticed that the gravitational action of luminous matter is not sufficient to hold clusters together (Zwicky 1933; Smith 1936). Rubin and others brought the problem closer to home by showing that spiral galaxies like our suffer the same problem (see e.g., Knapp & Kormendy, 1987). While the leading explanation for the dark matter problem today is slowly moving, weakly interacting “nonluminous” elementary particles remaining from the earliest moments – cold dark matter (see

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e.g., Turner, 2000) – there is still interest in the possibility that the explanation involves new gravitational physics (see e.g., Sellwood & Kosowsky, 2000).

Any gravitational explanation must deal with the fact that the shortfall of the Newtonian gravity of luminous matter occurs at widely different length scales – at distances less than 1 kpc in dwarf spirals to distances greater than 100 kpc in clusters of galaxies. Merely strengthening of gravity at a fixed distance cannot explain away the need for dark matter.

In 1983 Milgrom (1983a,b) made a remarkable observation: the need for the gravitational effect of nonluminous (dark) matter in galaxies only arises when the Newtonian acceleration is less than about $a_0 = 2 \times 10^{-8} \text{ cm s}^{-1} = 0.3 c H_0$. (Here, $H_0 = 70 \pm 7 \text{ km s}^{-1} \text{ Mpc}^{-1} = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ is present expansion rate of the Universe.) This fact is the foundation for his Modified Newtonian Dynamics (MOND) alternative to particle dark matter.

MOND can be described as a modification of Newton’s second law, with the Newtonian force law remaining the same, but no dark matter. In Newtonian theory, for a spherically symmetric distribution of matter the acceleration of a mass in a circular orbit and the gravitational force are related by,

$$a(r) = \frac{v^2(r)}{r} = \frac{GM(r)}{r^2} \quad (1)$$

where $M(r)$ is the mass interior to radius r . In MOND the left-hand side is modified by a function $\mu(a/a_0)$:

$$\mu(a/a_0) \cdot \frac{v^2(r)}{r} = \frac{GM_V(r)}{r^2} \quad (2)$$

where M_V denotes the luminous mass.

To eliminate the need for unseen (dark) matter in spiral galaxies $\mu(x)$ must have the following properties: $\mu(x) = 1$ for $x \gg 1$, and $\mu(x) = x$ for $x \ll 1$. As a consequence of $\mu(x) = x$ for $x \gg 1$, the circular rotation speed of a spiral galaxy asymptotes to a value v_∞ such that

$$v_\infty^4 = GM_V a_0, \quad (3)$$

where M_V is the total visible mass. Such an empirical relation for spiral galaxies is well known: the Tully – Fisher relation.

While MOND does away with the need for dark matter in a very clever way and neatly explains the empirical Tully – Fisher relationship, it lacks predictive power because it cannot be fit into a consistent relativistic theoretical framework (see e.g., Bekenstein & Milgrom, 1984, or Sanders, 1998). In essence, MOND is an empirical rule which seems to work well on galactic scales. It cannot address many of the most important and interesting phenomena in modern astrophysics and cosmology – gravitational lensing by galaxies and

clusters of galaxies, the striking successes of big-bang cosmology – big-bang nucleosynthesis (BBN), cosmic microwave background (CMB) anisotropy and structure formation through gravitational instability – and the physics of compact objects such as black holes and neutron stars.

A recent *ad hoc* attempt to use MOND (McGaugh 1999, 2000) to explain the surprising smallness of secondary acoustic peaks in the CMB angular power spectrum revealed in the early measurements of CMB anisotropy on sub-degree scales (Hanany et al 2000; Lange et al 2001) has turned into yet another observation that MOND cannot explain. New more precise measurements of fine-scale CMB anisotropy now clearly show second and third acoustic peaks in accord with the BBN baryon density and the gravitational driving force of particle dark matter (Pryke et al 2001; Netterfield et al 2001).

MOND has other problems too. While MOND successfully reproduces the baryonic mass–temperature relationship in clusters, it predicts a rising temperature profile which is in strong disagreement with measurements (Aguirre et al 2001). (We will have more to say about clusters later.) Ly α absorbers have characteristic accelerations less than a_0 and MOND should apply; Aguirre et al (2001) argue that the predicted sizes of Ly α absorbers are much smaller than observed. Finally, the SuperKamiokande (Fukuda et al, 1998) evidence for neutrino mass, based upon the zenith-angle dependence of the atmospheric muon neutrino flux, implies that relic, nonluminous neutrinos contribute about as much (or more) mass density than do luminous stars: nonluminous, particle dark matter exists.

A recent paper by Scott et al (2001) chronicles the many cosmological difficulties that MOND now faces, especially those associated the formation of structure in the Universe and the anisotropy of the cosmic microwave background.

While it is becoming ever harder to take MOND seriously even as a feature of a new, yet to be formulated gravitation theory, the empirical fact that the need for dark matter in galaxies always seems to occur at an acceleration of around cH_0 still stands. Any successful theory of structure formation must explain this fact. It is the purpose of this *Letter* to show how Milgrom’s Law arises quite naturally in the cold dark matter theory of structure formation.

2. How CDM Predicts Milgrom’s Law

2.1. CDM theory

The cold dark matter theory of structure formation has two basic features: seed density inhomogeneity that arose from quantum fluctuations during inflation and dark matter in the Universe exists in the form of slowly moving particles left over from the big bang. The two leading candidates for the CDM particle are the axion and the neutralino. Each is predicted by a compelling extension of the standard model of particle physics motivated by particle-physics considerations (rather than cosmological) and has a predicted relic density comparable to that of the known matter density (see e.g., Turner, 2000).

A recent estimate of the total matter density puts it at $\Omega_M = 0.33 \pm 0.035$ (Turner, 2001). The two most precise measurements of the total baryon density are based upon BBN and primeval deuterium measurements and CMB anisotropy measurements: $\Omega_B h^2 = 0.020 \pm 0.001$ (BBN; see Burles et al, 2001) and $0.023^{+0.004}_{-0.003}$ (CMB; see Pryke et al, 2001; Netterfield et al, 2001). Folding in current knowledge about the Hubble constant ($h = 0.72 \pm 0.07$; Freedman et al, 2001), implies $\Omega_B = 0.04 \pm 0.008$, and further, that CDM particles contribute $\Omega_{\text{CDM}} = 0.30 \pm 0.04$.

While we now know that neutrinos contribute as much to the universal matter density as do stars (Fukuda et al 1998), about 0.5% of the critical density, Croft et al (1999) argue based upon the formation of small-scale structure, that neutrinos can contribute no more than about 10% of the critical density. Thus, if the particle dark-matter explanation is correct, the bulk of the dark matter is CDM particles, which provide the gravitational infrastructure, with baryons providing the light.

For our purposes here, a less essential feature of CDM is the fact that the bulk of the critical density exists in the form of a mysterious dark energy ($\Omega_X \simeq 0.66 \pm 0.06$; see e.g., Turner, 2001). The simplest possibility for the dark energy is vacuum energy (mathematically equivalent to Einstein’s cosmological constant). The direct evidence for dark energy comes from supernova measurements (Riess et al, 1998; Perlmutter et al, 1999). Strong, indirect support comes from the fact that the CMB anisotropy measurements indicate a flat Universe, $\Omega_0 = 1.0 \pm 0.04$ (Netterfield et al 2001; Hanany et al 2000; Pryke et al 2001), and matter falls short of the critical density by a factor of three. While the existence of dark energy affects the details of structure formation enough so that observations can discriminate between a matter-dominated flat Universe and one with dark energy, for the purposes of showing how CDM predicts Milgrom’s Law, dark energy and its character are not critical. This is because most galaxies formed while the Universe was still matter-dominated and well described by the Einstein – deSitter model.

In the CDM scenario, structure forms from the bottom up, through hierarchically merging of small halos to form larger halos (see e.g., Blumenthal et al, 1984). The bulk of galactic halos formed around redshifts of 1 to 5, with clusters forming at redshifts of 1 or less, and superclusters forming today. Within halos, baryons lose energy through electromagnetic interactions and sink to the center, supported by their angular momentum (believed to have arisen through tidal torquing). Until baryonic dissipation sets in, baryons and CDM particles exist in a universal ratio of $\Omega_{\text{CDM}}/\Omega_B \simeq 7$. Were it not for the concentration of baryons caused by dissipation, the gravity of dark matter would be dominant everywhere.

2.2. CDM and Milgrom’s Law

The CDM explanation for the gravitational effect of dark matter “kicking in” at a fixed acceleration approximately equal to cH_0 involves three ingredients: i) the fact that the Universe is reasonably well described by the Einstein – deSitter model during the period when galaxies form; ii) the scale-free character of the seed density perturbations over the relevant scales; iii) baryonic dissipation; and iv) some (apparent) numerical coincidences.

The argument begins with facts i) and ii), which lead to the CDM prediction of self-similar dark-matter halos. Halos, regardless of their mass, can be described by the same mathematical form (Navarro et al, 1997). The exact functional form is not essential; for simplicity we write the halo profile for an object that began from perturbations of comoving length scale L as

$$\rho_L(r) \simeq \beta \Omega_M \rho_{\text{crit}} (1 + z_c)^3 (r/L)^{-2} \quad (4)$$

where $\rho_{\text{crit}} = 3H_0^2/8\pi G$ is the critical density today, z_c is the redshift of halo collapse and β is a numerical constant of $\mathcal{O}(100)$. Because $\Omega_M(1 + z_c)^3 \rho_{\text{crit}}$ is the mean matter density at the redshift of collapse (and also approximately the critical density at that epoch), Eq. 4 says that the mean density of the collapsed structure is about 100 times the ambient density when collapse occurred.

The redshift of collapse is related to the spectrum of density perturbations: collapse on length scale L occurs when the density contrast ($= \delta\rho/\rho$) on that scale is order unity. The density contrast (neglecting nonlinear effects) at redshift z is related to the matter power spectrum today:

$$\left(\frac{\delta\rho}{\rho}\right)_L \simeq \sqrt{\frac{k^3 |\delta_k|^2}{2\pi^2}} \simeq (\epsilon/10^{-5})(1+z)^{-1}(L/L_0)^{-\frac{1}{2}(n_{\text{eff}}+3)} \quad (5)$$

where $k \sim L^{-1}$, $n_{\text{eff}} \approx -2.3$ is the logarithmic slope of the power spectrum around galaxy scales (assuming the primeval density perturbations are scale invariant; see e.g., Bardeen

et al, 1986), ϵ is the dimensionless amplitude of the primeval fluctuations in the gravitational potential, determined by COBE (White and Bunn 1995) to be about 10^{-5} , and $L_0 \simeq 10h^{-1} \text{ Mpc}$ is the scale of nonlinearity today (for $\epsilon \sim 10^{-5}$). Substituting Eq. 5 into Eq. 4, it follows that

$$\rho_L(r) = [(3\beta/8\pi) \Omega_M (\epsilon/10^{-5})^3] (H_0^2/G) (L/L_0)^{-\frac{3}{2}(n_{\text{eff}}+3)} (r/L)^{-2} \quad (6)$$

The third ingredient is baryonic dissipation: after halos form, their baryons dissipate energy and collapse in linear scale by a factor $\alpha \approx 10$ to form a disk supported by angular momentum (for work on modelling disk collapse, see e.g., Dalcanton et al, 1997; Mo et al, 1998). Because of the increased concentration of baryons interior to $r \sim L/\alpha$, their gravity will dominate the dynamics in the inner regions. Thus, the transition from dark-matter dominated gravity to luminous-matter dominated gravity should occur at $r_{\text{DM}} = L/\alpha \sim L/10$. The acceleration at this point is

$$a_{\text{DM}} \equiv a(r_{\text{DM}}) = \frac{GM(r_{\text{DM}})}{r_{\text{DM}}^2} = [4\pi G (L/\alpha)] \rho_L(L/\alpha)$$

After some re-writing, Milgrom’s Law appears

$$\begin{aligned} a_{\text{DM}} &= cH_0 \left[\frac{3}{2} \beta \alpha \Omega_M \left(\frac{\epsilon}{10^{-5}} \right) \right] (c^{-1} H_0 L_0) \left(\frac{L}{L_0} \right)^{-\frac{3}{2}n_{\text{eff}} - \frac{7}{2}} \\ &= \mathcal{O}(1) cH_0 \left(\frac{L}{L_0} \right)^{-0.05} \end{aligned} \quad (7)$$

where we have used $n_{\text{eff}} = -2.3$. The final ingredient is the conspiracy of numerical factors to give a coefficient of unity and a very mild scale dependence, $L^{-0.05}$.

The mild scale dependence of the acceleration where dark matter dominates owes to the fact that $n_{\text{eff}} \approx -2.3 \approx -\frac{7}{3}$, which only holds around galactic scales. It arises from a combination of the primeval spectral index ($n \simeq 1$) and the bending of the shape of the spectrum of perturbations caused by the fact that perturbations on small scales ($k \gtrsim 0.1 \text{ Mpc}^{-1}$) entered the horizon when the Universe was radiation-dominated and those on large scales ($k \lesssim 0.1 \text{ Mpc}^{-1}$) entered the horizon when the Universe was matter-dominated (see e.g., Bardeen et al, 1986). For $k \ll 0.1 \text{ Mpc}^{-1}$, $n_{\text{eff}} \rightarrow 1$ and for $k \gg 0.1 \text{ Mpc}^{-1}$, $n_{\text{eff}} \rightarrow -3$.

Returning to the numerical conspiracy that leads to $a_{\text{DM}} \sim cH_0$; for $n_{\text{eff}} = -\frac{7}{3}$, the factor $(\epsilon/10^{-5})^3 L_0$ is just the scale of nonlinearity today, independent of the actual value of ϵ . The numerical coincidence then is the fact that the scale of nonlinearity today is much less than the Hubble scale. Scott et al (2001) have attempted to tie this fact to the cooling scale of baryons, which can be related to fundamental constants and ϵ .

Equation 7 only holds around galaxy scales ($L \sim 1$ Mpc), where $n_{\text{eff}} \approx -\frac{7}{3}$ and $\alpha \sim 10$. Because of this, the MOND prescription does not work for clusters. First, baryons do not dissipate significant and sink to the center, and thus, they are everywhere dark matter dominated. This, in spite of the fact that for clusters, $a \lesssim 10^{-8} \text{ cm s}^{-2}$. Said another way, CDM correctly predicts that Milgrom’s Law *should not* apply to clusters.

The issue of the shape of the halo density profile is not central to our arguments. Regardless of the shape of the profile, it will be true that $a_{\text{DM}} \propto L(1+z_c)^3$ which results in the very mild scale dependence when Eq. 5 is used.

The derivation of Eq. 7 is the key result of this paper. We have shown that Milgrom’s Law – the need for dark matter in galaxies at accelerations less than about cH_0 – is predicted by CDM. While scale-free density perturbations, an epoch where the Universe is well described by the Einstein – deSitter model and baryonic dissipation are essential, the fact that a_{DM} is nearly cH_0 appears to be a numerical coincidence. Furthermore, a_{DM} is a fixed number since galaxies are bound and well relaxed today, while cH decreases with time. Thus, the approximate equality of a_{DM} with cH only holds today.

Another coincidence for CDM is known. The galaxy-galaxy correlation function is very well fit by a power law, $\xi(r) = (r/r_0)^{-1.8}$ where $r_0 = 5h^{-1}$ Mpc (see e.g., Groth & Peebles, 1977; Baugh, 1996). In CDM theory, the two-point correlation function of mass is not a good power law; however, when bias is taken into account (the nontrivial relation between mass and light), the galaxy – galaxy correlation function turns into a power-law (see e.g., Pearce et al, 1999), in good agreement with observations.

3. Concluding remarks

By any measure, CDM is the most expansive and most successful theory of structure formation yet proposed. It is motivated by compelling particle physics (the neutralino and the axion) and inflationary cosmology (nearly scale-invariant, adiabatic density perturbations). It is very predictive and has already passed many important tests. CDM is not without a few problems (see e.g., Sellwood & Kosowsky, 2000), the most nagging of which is the problem of cuspy, clumpy halos which *seems* to be at variance with observation (see e.g., de Blok et al, 2001); Borriello & Salucci, 2001).

The most fundamental elements of CDM are now being tested by accelerator searches for neutralinos, specialized detectors searching for halo dark matter, and measurements of large-scale structure and CMB anisotropy. Time will tell how much of the cosmic truth CDM holds.

The emergence of another gravitational mystery, the dark energy that powers the accelerating Universe, has renewed interest in a gravitational solution that explains both dark problems. Whether or not Milgrom’s Law is a clue to such a solution remains to be seen. It is, however, an empirical fact that any complete theory of structure formation must explain. In this *Letter* we have shown how Milgrom’s law arises in the cold dark matter theory of structure formation.

Separating important clues from misleading coincidences is at the heart of scientific creativity. Hoyle’s observation that the energy released in burning 25% of the Hydrogen to Helium is approximately equal to that of the CMB suggested a non big-bang origin for the CMB (see e.g., Burbidge & Hoyle, 1998). On closer examination, the agreement is too good! Using the BBN/CMB baryon density ($3.8 \times 10^{-31} \text{g cm}^{-3}$) and assuming a Helium mass fraction of 24% leads to a energy density produced from proton fusion that is only 20% more than that of the CMB. This implies that the Helium must have been produced at a mean redshift of about 0.2, leaving essentially no time to thermalize the starlight and making problematic the large abundance of Helium seen in high redshift quasars. Hoyle’s observation appears to be a misleading coincidence. Unlike Milgrom’s Law, no explanation for the coincidence (within the big-bang model) has yet been put forth.

Within the next decade the particle dark-matter hypothesis will be put to the test, and we should settle the question of whether or not Milgrom’s Law is an important clue or, as it now appears, a misleading coincidence.

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